Pre-class Warm-up!!!

What is $\dfrac{\partial^2 f}{\partial x \partial z}$ when $f(x,y,z) = xy + z\sin(x)$?

✓ a. $\cos(x)$

b. $y + \sin(x)$

c. $z\cos(x)$

d. y

e. None of the above

$\dfrac{\partial^2 f}{\partial x \partial z}$ means $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right)$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial z}\left(y + z\cos x\right)$$

$$= \cos x = \frac{\partial^2 f}{\partial x \partial z}$$

3.1 Iterated partial derivatives

We have already been using these in showing that curl(grad(f)) = (0, 0, 0) and div(curl(F)) = 0.

In Section 3.1 they define them and prove symmetry of the mixed partial derivatives using the mean value theorem.

HW questions are all: calculate these mixed partial derivatives, verify that this function satisfies this partial differential equation.

3.2 Taylor's theorem.

Question. What is the Taylor expansion of $f(x) = x^2$ about $x = 1$?

a. $f(x) = 1 - 2(x+1) + (x+1)^2$

✓ b. $f(x) = 1 + 2(x-1) + (x-1)^2$

c. $f(x) = 1 - 2(1-x) + (1-x)^2$

✓ d. $f(1+x) = 1 + 2x + x^2$

e. None of the above.

The Taylor expansion of $f(x) = x^2$ about $x = 0$ is x^2

We learn:

- What Taylor polynomials are.
- What a Taylor series is.
- What Taylor approximations are.
- The form of the terms in a Taylor polynomial.
- Taylor's theorem
- How to write the degree 1 and 2 polynomials in terms of the gradient and the Hessian matrix. derivative

We don't need to know

- The forms of the remainder terms

The Taylor expansion of $f(x)$ about $x = 1$ has the form

$$f(1+h) = a_0 + a_1 h + a_2 h^2 + \dots$$

$$f(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots$$

The form of the coefficients in the Taylor series (1-variable case)

Do this first for the expansion about 0:

f(x) = a_0 + a_1 x + a_2 x^2 +

We can compute the numbers $a_0, a_1, \dots$

$f(0) = a_0$

Apply $\frac{d}{dx}$ to both sides

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

Put $x = 0$: $f'(0) = a_1$

Apply $\frac{d}{dx}$: $f^{(2)}(x) = 2a_2 + 3 \cdot 2 a_3 x + \dots$

Put $x = 0$ $\quad a_2 = \frac{1}{2} f^{(2)}(0)$

$a_n = \frac{1}{n!} f^{(n)}(0)$.

Next: the expansion about c :

f(c+h) = a_0 + a_1 h + a_2 h^2 +

Do the same but evaluate when $c+h = c$ i.e. $h = 0$.

$f(c) = a_0$

Apply $\frac{d}{dx}$: $f'(c+h) = a_1 + 2a_2 h + \dots$

At $h = 0$ $\quad f'(c) = a_1$

etc.

$a_n = \frac{1}{n!} f^{(n)}(c)$.

The expansion about c :

$f(c+h) = a_0 + a_1 h + a_2 h^2 + \dots$

$= \left(a_0 + a_1 h + \dots + a_2 h^n\right) + R_n(c,h)$

= Taylor polynomial of degree n
+ Remainder term of degree n

where $a_i = \frac{1}{i!} f^{(i)}(c)$

Taylor's theorem:
$R_n(c,h) / h^n \to 0$ as $h \to 0$

When $n = 1$ we get

$f(c+h) = f(c) + f'(c)\, h + R_1(c,h)$

$f(c) + f'(c)h$ is the best linear approximate to f around c.

Also $\frac{R_1(c,h)}{h} = \frac{f(c+h) - f(c) - f'(c)h}{h}$

$\to 0$ as $h \to 0$

is the definition of the derivative (equivalently).

Taylor series with more than one variable

Now f : R^n -> R
and c = (c_1,...,c_n), h = (h_1,...,h_n) lie in R^n

The Taylor series about c has the form

f(c+h)= $a_{0\cdots0} + a_{10\cdots0}h_1 + a_{010\cdots0}h_2 + \cdots + a_{0\cdots01}h_n$

$+ a_{20\cdots0}h_1^2 + a_{020\cdots0}h_2^2$

$+ a_{110\cdots0}h_1h_2 + a_{1010\cdots0}h_1h_3 + \cdots$

$+ a_{30\cdots0}h_1^3 \cdots$

e.g. $h_2^3 h_4^2$ has coeff $a_{03020\cdots0}$

What are the coefficients a ?

Evaluate at $\underline{h}=(0,0,0\ldots)$ and then apply $\frac{\partial}{\partial x_j}$. Repeat.

$f(c) = a_{0\cdots0}$ Apply $\frac{\partial f}{\partial x_1}(c+h) = a_{10\cdots0} + 2a_{20\cdots0}h_1 + a_{110\cdots0}h_2 + \cdots$

At $h=0$: $a_{10\cdots0} = \frac{\partial f}{\partial x_1}(c)$

$$a_{i_1\cdots i_n} = \frac{1}{i_1!}\frac{1}{i_2!}\cdots\frac{1}{i_n!}\frac{\partial^{i_1}}{\partial x_1^{i_1}}\cdots\frac{\partial^{i_n}}{\partial x_n^{i_n}}f(c).$$

Example: n = 2

$$f(x+h_1, y+h_2) = \underbrace{f(x,y)}_{\text{Constant}} + \underbrace{\frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}\cdot h_2}_{\text{linear terms}}$$

$$+ \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}h_1^2 + \frac{\partial^2 f}{\partial x\partial y}h_1h_2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}h_2^2}_{\text{quadratic terms}}$$

$+$ higher terms.

$$= f(x,y) + Df(x,y)(h_1,h_2) + \frac{1}{2}\underline{h}^T H \underline{h} + \text{higher terms}$$

We see the best linear approximation and the best quadratic approximation to f around c.

The Hessian matrix H

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1 \partial x_3} & \cdots \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial f}{\partial x_2 \partial x_3} & \\ \vdots & & & \end{bmatrix}$$

$$[h_1, \ldots, h_n] \, H \begin{bmatrix} h_1 \\ \\ h_n \end{bmatrix}$$ is 2x the quadratic part of the Taylor expansion

Example: Find the first and second degree Taylor polynomials and the Hessian matrix for f(x,y) = sin(xy) at c = (1, π/2). Use these to approximate f(1.1,π/2).